Appendix A  

The Theory of Optimal Betting Spreads

Introduction

The purpose of this appendix is to show that the theory of optimal betting spreads for the game of Blackjack has a sound mathematical basis, and to clear up confusion resulting from inadequate or non-existent definitions of the terms which are used. Furthermore, it will be demonstrated that the choice of an optimal spread, is independent of the choice of betting method, be that a proportional Kelly scheme, or a fixed scheme with a known risk of ruin. In addition, a method is presented which will enable direct calculation of the optimal spread for a given game, choice of maximum bet, and optional (Wonging) points at which no bet is made.

In Section 1, the mathematical definitions of betting spreads and betting schemes, are presented, which allow a precise formulation of the problem. The other parameters such as system expectation, variance are also defined, as well as the functional $n_0$ which plays a central role in the analysis. In Section 2, the well-defined mathematical problem of finding the spread (as defined in Section 1) which minimises the functional $n_0$, subject to the constraints, is examined. The form of the solution is found, and a method of direct construction is presented. In Section 3, there is a digression into Kelly betting theory, where it is shown that subject to the appropriate expansion of the Log utility function, that the minimisation of $n_0$ is a part of the computation of the optimal Kelly betting scheme. In Section 4, the problem is approached from the perspective of a fixed betting scheme, where in this case, instead of choosing a bet size based on our current bankroll as in the Kelly scheme, bet size must be chosen to give a desired risk of ruin for a particular bankroll. Further it is shown, that for a fixed risk of ruin, the spread which minimises $n_0$ is the spread which gives the fastest linear bankroll growth. Section 5 presents an example of a game with specified parameters, and using the results of Section 2, a direct calculation of the optimal spread is performed. To illustrate the distinction between this procedure and the betting scheme used, separate examples of the use of this spread for a Kelly and a fixed bettor are given. Finally there is a summary of the results and main conclusions of the article.

Definitions

In order to define the problem we require precise definitions of the terms which are used. Firstly we define a Game as the countable set, one which may be labelled by integer values,

\[ i \text{ count index}, \]
\[ e_i \text{ unit expectation}, \]
\[ v_i \text{ unit variance}, \]
\[ c_i \text{ frequency of 'count index' for the game}, \]

where the index $i$ may range over all integer values $-\infty < i < \infty$. The values of $e_i$, $v_i$ and $c_i$ are real (non-integer), and subject to the requirements,

\[ -\infty < e_i < +\infty \]
\[ v_i \geq 0 \]
\[ \sum_{i=-\infty}^{\infty} c_i = 1 \]  \hspace{1cm} (1)

and $c_i \to 0$ for large $|i|$ sufficiently fast for all subsequently defined sums to be convergent.
Since \([i]\) is a countable set, it is possible without loss of generality to order the set of values such that
\[
e_{i+1}/v_{i+1} \geq e_i/v_i
\]
(2)
for all \(i\).

Define a **Betting Spread** to be a set of values \(b_i\) for each \(i\), such that
\[
b_i = 0 \quad \forall i \in \{I_{Wong}\}
\]
\[
b_i = 1 \quad \forall i \in \{I_{Min}\}
\]
\[
b_i = M \quad \forall i \in \{I_{Max}\}
\]
\[
1 < b_i < M \quad \forall i \in \{I_{Inter}\}
\]
(3)

Note, with the above definition, there can be no negative values of \(b_i\), it is bounded above by \(M\), and may not assume any values in the interval \((0,1)\). Also, there are no units monetary or otherwise, associated with the spread \(b_i\).

In order to apply \(b_i\) to a real game of Blackjack, we will later require the definition of an associated **Betting Scheme** \(B_i\) which is
\[
B_i = B \cdot b_i \quad \forall i
\]
(4)

where \(B\) is a **positive** real number with units of currency (Dollars, Pounds etc.). A convention is used here whereby all quantities which refer to the unit game are given in lower case, whereas those in an associated monetary system using a betting scheme are given in upper case.

Now, given a specified **Game**, and betting spread \(b_i\), we can define the functionals
\[
ev[b] = \sum_{i \in \{I_{Wong}\}} b_i \cdot c_i \cdot e_i
\]
\[
var[b] = \sum_{i \in \{I_{Wong}\}} b_i^2 \cdot c_i \cdot v_i
\]
(5a)
(5b)

and the **Long Run Index**
\[
n_0[b] = \frac{var[b]}{ev^2[b]}
\]
(6)

The functional \(n_0[b]\) is called the **Long Run Index**, because it is equal to the number of rounds that must be played, with a fixed betting spread, such that the accumulated expectation equals the accumulated standard deviation. As such, it is a measure of how many rounds must be played to overcome a negative fluctuation of one standard deviation with a fixed spread.
Postulate:

An Optimal Betting Spread for a Game, is the Betting Spread $b_i$ such that $n_0[b]$ is minimised, subject to the constraints of maximum value $M$, the specified set $\{I_{\text{Wong}}\}$, and the requirement that $\text{ev}[b] > 0$.

At this stage, this is a postulate based upon the judgement that it is desirable for the expectation to outgrow the standard deviation as quickly as possible for a fixed betting spread. Clearly the condition $\text{ev}[b] > 0$ is self-evident, as we only consider games and spreads which have a positive net expectation. It will be shown later that an optimal spread defined in this way, is exactly the spread which gives the fastest bankroll growth in a proportional Kelly system, subject to the usual Log expansion.

Section 2 - Minimisation of $n_0[b]$

At this point we have a well defined mathematical problem:

For a given value of $M$, specified zero set $\{I_{\text{Wong}}\}$, and requirement $\text{ev}[b] > 0$, find the optimal spread $b_i^{opt}$, which minimises $n_0[b]$. Firstly, since $\{I_{\text{Wong}}\}$ is a countable subset, it is possible to remove this subset from the ordered set of values $\{\text{Game}\}$, and the result is still a Game, except the values of $c_i$ no longer add to unity. That is, we no longer play 100% of all hands dealt. Since this requirement is not necessary for the result, we will assume without loss of generality, that all zero values $\{I_{\text{Wong}}\}$ have been removed and that the minimum possible value of $b_i$ is now unity.

Since $n_0[b]$ is a function of all the $b_i$, if

$$
\frac{dn_0[b]}{db_i} = \left[ \frac{d \text{var}[b]}{db_i} \cdot \text{ev}[b] - 2 \frac{d \text{ev}[b]}{db_i} \cdot \text{var}[b] \right] / \text{ev}^3[b] 
$$

$$
= 0
$$

(7)

where

$$
\frac{d \text{ev}[b]}{db_i} = c_i \cdot e_i
$$

(8)

$$
\frac{d \text{var}[b]}{db_i} = 2b_i \cdot c_i \cdot v_i
$$

(9)

then Eqn(7) is satisfied if,

$$
b_i \cdot v_i \cdot \text{ev}[b] - e_i \cdot \text{var}[b] = 0
$$

or,

$$
b_i = \frac{e_i \cdot \text{var}[b]}{v_i \cdot \text{ev}[b]} 
\equiv \frac{e_i}{v_i} \cdot e^k[b]
$$

(10)
where \( e_{kb}[b] \) is the system functional defined as

\[
e_{kb}[b] = \frac{var[b]}{ev[b]}.
\]  

(11)

Note that Eqn(10) is a highly non-linear system, since \( e_{kb}[b] \) depends on the values of all the \( b_i \), including the particular value on the left hand side. Eqn(10) can be viewed as a consistency condition for a spread, once it has been found. It also has to be remembered that it is not possible to satisfy Eqn(10) simultaneously for all values of \( \{i\} \), since \( b_i \) is constrained to be between 1 and \( M \), and be such that \( ev[b] > 0 \).

Since the values of \( e_i/v_i \) are ordered, and \( e_{kb}[b] > 0 \) (since \( ev[b] > 0 \)), we arrive at a constraint upon the values of \( i \) for which Eqn(10) can be satisfied. Since \( b_i \) is bounded between 1 and \( M \), we have

\[
1 < \frac{e_i}{v_i} \cdot e_{kb}[b] < M
\]  

(12)

If there is some set of values \( i \) for which Eqn(12) is satisfied, there will be a maximum value of \( e_{T_{M-1}}/v_{T_{M-1}} \) in the set. Since the set is ordered, all values of \( i \) greater than or equal to \( T_M \) will fail to satisfy Eqn(12). Similarly, the minimum value \( e_{T_{1+1}}/v_{T_{1+1}} \) defines a lower limit such that for all \( i \) less than or equal to \( T_1 \) will also fail Eqn(12) and by definition \( T_1 < T_M \).

Construct a semi-optimal betting spread \( b_i \), such that

\[
b_i = 1 \quad i \leq T_1
\]

\[
b_i = \frac{e_i}{v_i} \cdot e_{kb}[b] \quad T_1 + 1 \leq i \leq T_M - 1
\]  

(13)

\[
b_i = M \quad i \geq T_M
\]

where again, \( e_{kb}[b] \) is a function of all the \( b_i \). Equation (13) is a circular definition, nevertheless the family of spreads so defined, form a subset of all possible spreads due to the constrained nature of the intermediate values. It will shortly be shown that for spreads of this form, the value of \( e_{kb}[b] \) can be determined in a consistent way, and the whole spread may be calculated.

For spreads of the form Eqn(13), the system functionals may be re-expressed as

\[
ev[b] = ev_-[T_1] + e_{kb}[b] \cdot ev_{\text{int}}[T_1, T_M] + M \cdot ev_+[T_M]
\]  

\[
\text{var}[b] = \text{var}_-[T_1] + e_{kb}^2[b] \cdot \text{var}_{\text{int}}[T_1, T_M] + M^2 \cdot \text{var}_+[T_M]
\]  

(14a)

(14b)

where

\[
ev_-[T_1] = \sum_{i=\infty}^{T_1} c_i \cdot e_i
\]  

(15a)

\[
ev_{\text{int}}[T_1, T_M] = \sum_{i=T_1+1}^{T_M-1} c_i \cdot \frac{e_i^2}{v_i}
\]  

(15b)

\[
ev_+[T_M] = \sum_{i=T_M}^{\infty} c_i \cdot e_i
\]  

(15c)
and
\[ \text{var}_i[T_i] = \sum_{i=-\infty}^{T_i} c_i \cdot v_i \]  
(16a)
\[ \text{var}_{int}[T_1, T_M] = \sum_{i=T_1+1}^{T_M-1} c_i \cdot \frac{v_i^2}{v_j} \]  
(16b)
\[ \text{var}_a[T_M] = \sum_{i=T_M}^{\infty} c_i \cdot v_i \]  
(16c)

Note, that for spreads of the form Eqn(13)
\[ \text{ev}_{int}[T_1, T_M] = \text{var}_{int}[T_1, T_M] = D[T_1, T_M] \]  
(17)
and so
\[ \text{ev}[b] = \text{ev}_i[T_i] + ekb[b] \cdot D[T_1, T_M] + M \cdot \text{ev}_a[T_M] \]  
(18a)
\[ \text{var}[b] = \text{var}_i[T_i] + ekb^2[b] \cdot D[T_1, T_M] + M^2 \cdot \text{var}_a[T_M] \]  
(18b)
But we also have the definition Eqn(11), which when combined with Eqns(18a) and (18b) gives
\[ ekb[b] \cdot \text{ev}_i[T_i] + ekb^2[b] \cdot D[T_1, T_M] + ekb[b] \cdot M \cdot \text{ev}_a[T_M] 
= \text{var}_i[T_i] + ekb^2[b] \cdot D[T_1, T_M] + M^2 \cdot \text{var}_a[T_M] \]  
(19)
where the terms involving \( D[T_i, T_M] \) cancel. Equation(19) can be immediately solved for \( ekb[b] \), to give
\[ ekb[b] = \frac{\text{var}_i[T_i] + M^2 \cdot \text{var}_a[T_M]}{\text{ev}_i[T_i] + M \cdot \text{ev}_a[T_M]} \]  
(20)
Equation(20) is one of the most important equations in optimal spread theory. It says that for spreads of the form Eqn(13) the value of \( ekb[b] \) is completely determined by the subsets \( \{I_{\text{Min}}\} \) and \( \{I_{\text{Max}}\} \). But by construction, spreads of the form Eqn(13) satisfy Eqn(10) for all values of \( i \) between \( T_1 \) and \( T_M \), and are therefore of the form required to minimise \( n_M[b] \). Also since we have the constraint that \( \text{ev}[b] \) must be positive, \( ekb[b] \) must also be positive by Eqn(11). This leads to the observation that if \( \text{ev}_i[T_i] < 0 \) (which it almost always is unless backcounting) and for some \( T_M \), we must have
\[ M > \text{MAX} \left\{ \frac{\text{ev}_i[T_i]}{\text{ev}_a[T_M]}, 1 \right\} \]  
(21)
which is an absolute constraint on the minimum spread required in this Game (assuming of course \( \{I_{\text{Wong}}\} \) has been excluded), for a given \( T_1 \) and \( T_M \). Of course if \( \text{ev}_i[T_i] \geq 0 \) then the minimum value of \( M \) is 1, and the Game has a positive expectation with flat betting.

However, for an arbitrary pair of values \( (T_i, T_M) \), the value of \( ekb[b] \) given by Eqn(20) may not necessarily specify a 1 to \( M \) spread. If some intermediate \( b_i < 1 \) or \( b_i > M \), we will have a ratio of maximum to minimum value greater than \( M \), which is not a \( 1-M \) spread, and as such cannot be considered as a solution. This provides a constraint on the values of \( T_i \) and \( T_M \) used in Eqn(20). Again, by the ordering operation, if \( b_i > 1 \), then \( b_j > 1 \), for all \( j > i \). It is therefore sufficient if
\[ b_{T_{i+1}} > 1 \]  
(22a)

or

\[ ek[b] > \frac{v_{T_{i+1}}}{e_{T_{i+1}}} \]  
(22b)

where it is also noted that

\[ e_{T_{i+1}} > 0 \]  
(22c)

is a necessary condition for Eqn(22a). Similarly, it is sufficient if

\[ b_{T_{M-1}} < M \]  
(23a)

or

\[ ek[b] < M \cdot \frac{V_{T_{M-1}}}{e_{T_{M-1}}} . \]  
(23b)

Note also that since \( e_{T_{i+1}} > 0 \), and \( T_{M} > T_{I} \), then \( e_{T_{M-1}} > 0 \).

Since the functionals defined in Eqns(15) and (16) depend only on the specified Game parameters \( c_{i}, e_{i}, \) and \( v_{i} \), and the chosen values of \( T_{I} \) and \( T_{M} \), it is a simple matter to compute the value of \( ek[b] \) defined by Eqn(20), and by inspection, exclude all values of \( T_{I} \) and \( T_{M} \) which fail to satisfy Eqns(22) and (23). Equation(22c) provides the most rigid bound on the value of \( T_{I} \), which may prove useful when designing a practical algorithm.

Provided Eqns(22) and (23) are satisfied, we have a \( I-M \) spread, which satisfies Eqn(10) for all \( i \) between \( T_{I} \) and \( T_{M} \), and since we know \( ek[b] \) we can compute the \( b_{i} \). However, we are not yet done. Since Eqn(10) is satisfied only for a subset of \( i \) values, \( n_{0}[b] \) may not necessarily be minimised, subject to the original constraints.

In order to determine if our spread defined by Eqn(13) and satisfying Eqns(22) and (23), is in fact the spread which minimises \( n_{0}[b] \), we need to examine the behaviour of \( n_{0}[b] \) around this point. At the minimum of \( n_{0}[b] \), it is necessary that any variation \( \delta b_{i} \) subject to the constraints, will act to increase \( n_{0}[b] \). For any i,

\[ b_{i} \rightarrow b_{i} + \delta b_{i} \]

then

\[ n_{0}[b] \rightarrow n_{0}[b] + \delta n_{0}[b] \]

and at the minimum we require

\[ \delta n_{0}[b] \geq 0 . \]  
(24)

If Eqn(24) is not satisfied then we have found a new spread \( b'_{i} \) for which \( n_{0}[b'] \) has a smaller value, and hence the original spread \( b_{i} \) was not the minimum solution. We have three cases to consider, since the constraints are different in each case.
Case(1): $i \leq T_i$

As $b_i = 1$, we must have $\delta b_i > 0$, since if $\delta b_i < 0$, we would require $b_i' < 1$ and the modified set $b_i$ would not be a spread. Using Eqn(7) and substituting $b_i = 1$, we have

$$\delta n_0[b] = \frac{dn_0[b]}{db_i} \cdot \delta b_i = \frac{2c_i}{ev^3[b]} \left[ (1) \cdot v_i \cdot ev[b] - e_i \cdot var[b] \right] \cdot \delta b_i \geq 0$$

or

$$v_i \cdot ev[b] - e_i \cdot var[b] \geq 0$$

which gives the condition

$$\frac{ev[b]}{var[b]} = \frac{1}{ekb[b]} \geq \frac{e_i}{v_i} \quad \forall i \leq T_i, \quad (25)$$

where due care has been taken with the inequalities, since it is possible $e_i$ may be less than zero. Again since the set is ordered, it is sufficient if

$$\frac{1}{ekb[b]} \geq \frac{e_i}{v_i} \quad . \quad (26)$$

Case(2): $i \geq T_M$

In this instance $b_i = M$ and $\delta b_i < 0$, since if $\delta b_i > 0$, we again would not have a spread. Using Eqn(7), except this time with $b_i = M$, we arrive at

$$M \cdot v_i \cdot ev[b] - e_i \cdot var[b] \geq 0,$$

which gives the analogous condition to Eqn(25),

$$\frac{ev[b]}{var[b]} = \frac{1}{ekb[b]} \leq \frac{1}{M} \cdot \frac{e_i}{v_i} \quad \forall i \geq T_M \quad . \quad (27)$$

and it again sufficient if

$$\frac{1}{ekb[b]} \leq \frac{1}{M} \cdot \frac{e_i}{v_i} \quad . \quad (28)$$
Case(3):  \[ T_i + 1 \leq i \leq T_M - 1 \]

For the intermediate case, \( \delta \) may be of either sign since \( 1 < \delta_i < M \). Therefore it is possible to increase or decrease the \( \delta_i \) and still have a 1-M spread. By the construction of Eqn(13), any intermediate \( \delta_i \) satisfies Eqn(7). Therefore it is required is to show that the second derivative is positive, so that the stationary point of \( n_0(\delta_i) \) is a minimum, not a maximum or a point of inflexion. The algebra is rather tedious, the reader is invited to fill in the gaps to show

\[
\frac{d^2 n_0(\delta_i)}{d\delta_i^2} = \frac{2c_i v_i}{ev^i[\delta_i]} \left[ ev[\delta_i] - c_i \cdot \delta_i \cdot e_i \cdot var[\delta_i] \right] = \frac{2c_i v_i}{ev^i[\delta_i]} \left[ ev[T_i] + M \cdot ev[N] \right]
\]

By Eqn(21), the first term must be positive. The second term can only be positive or zero, since from Eqns(15)-(17), \( D[T_1,T_M] > 0 \) and the subsequent term is included in the sum defining \( D[T_1,T_M] \). Therefore

\[
\frac{d^2 n_0(\delta_i)}{d\delta_i^2} \geq 0 \quad \forall \quad T_i + 1 \leq i \leq T_M - 1.
\]

Taking the reciprocal of Eqns(22b) and (23b), and combining them with Eqns(26) and (28), the final conditions are obtained,

\[
\frac{e_i}{v_i} \leq 1
\]

and

\[
\frac{1}{M} \cdot \frac{e_i}{v_i} \leq 1
\]

Equations (31) and (32) solve the problem as outlined. The procedure to find the optimal spread is reduced to finding the value of \( (T_i,T_M) \) for which \( ekb[b] \) as given by Eqn(20), satisfies Eqns(31) and (32). This may be done by inspection or by a computational algorithm. Once the correct value of \( ekb[b] \) is found, the spread may be quickly generated using Eqn(13).

An observation can be made here. Equations (31) and (32) impose a 'continuity' condition on spreads of the form Eqn(13). In other words,

\[
ekb[b] \cdot (e_{i}/v_{i}) < 1 \quad \text{and} \quad ekb[b] \cdot (e_{i+1}/v_{i+1}) > 1,
\]

and so there are no ‘jumps’ in the value of \( \delta_i \). Eqn(32) provides a similar condition at \( T_M \), so there is a smooth transition to the maximum bet. These conditions allow the solution to be written in a much more general form.

Combining Eqn(13) with Eqns(31) and (32), gives a spread of the form
\[
\begin{align*}
  b_i &= \frac{e_i \cdot ekb[b]}{v_i} \quad \text{if} \quad \frac{e_i \cdot ekb[b]}{v_i} \leq 1 \\
  b_i &= M \quad \text{if} \quad \frac{e_i \cdot ekb[b]}{v_i} \geq M
\end{align*}
\]

and
\[
  b_i = \frac{e_i \cdot ekb[b]}{v_i} \quad \text{otherwise},
\]

which no longer has any explicit ordering requirements on the values of \( e/v_i \).

A semi-optimal spread of the form Eqn(13), with a minimum of 1 unit and a maximum of \( M \) units, will automatically satisfy Eqns (11) and (20). That is, the ratio \( ev/\text{var} \) given in Eqn(11) for the entire Game, is equal to \( ev/\text{var} \) given in Eqn(20) for the game restricted to the maximum and minimum values. Also it can be seen from Eqns(14) and (17), that the \( ev/\text{var} \) for the game restricted to intermediate values is also equal to the same quantity \( ekb[b] \). This means that the constrained bets have the same risk as the intermediate, non-constrained bets, and when the two are combined, the total risk remains unchanged.

This is a necessary condition for optimality, but it is not sufficient. By choosing different values of \( T_1 \) and \( T_M \) in Eqn(20), many different semi-optimal spreads may be found. However, only one will satisfy Eqns (31) and (32), in which the spread makes a smooth transition from constrained to non-constrained bets at either end of the intermediate range. This spread is the optimal spread, since it minimises \( n_0 \) and therefore minimises the time to get into the long run. Equivalently the optimal spread minimises the doubling time for the Game.

A method for calculating the optimal spread without the need to specify of \( T_1 \) and \( T_M \) is the following:

1) Choose a non-optimal spread of the form Eqn(33) by replacing \( ekb[b] \) with a first guess \( ekb' \).
2) Compute \( ekb[b] \) from Eqns (5a),(5b) and (11).
3) Set the guess \( ekb' = ekb[b] \) and repeat the above steps until \( ekb[b] \) converges to a fixed value.

For all reasonable guess values, the procedure converges very quickly to the optimal solution. Since \( ekb' = ekb[b] \), the convergent solution satisfies Eqn(33) and also Eqn(12).

This procedure may be implemented in a spreadsheet program without much difficulty. By computing all the system functionals, Eqns (5), (6) and (11) for the guess \( ekb' \), and defining
\[
\delta ekb = ekb[b] - ekb',
\]
the solver function can be used to adjust the \( ekb' \) to solve \( \delta ekb = 0 \), subject to the constraints, \( ev[b] > 0 \) and \( \text{var}[b] > 0 \).

Note that at this point, we are talking about betting spreads \( b_i \), which do not have any monetary units. Moreover, the quantity \( ekb[b] \) is yet to be given any concrete interpretation. In order to use the above theory in the real world, a betting method such as a Kelly or fixed scheme must be chosen. Once such a scheme has been chosen, it becomes possible to directly interpret \( ekb[b] \) and the interpretation will depend on such a choice.

**Section 3 - Kelly Theory**

It is now necessary to diverge and discuss Kelly theory. Winston Yamashita independently derived an optimal betting criterion starting from the Kelly utility function. I will show that the minimisation of \( n_0[b] \) described above, plus an interpretation of the \( ekb[b] \), is equivalent to Yamashita theory. The Kelly Generalization method seeks to find the optimal Kelly betting scheme \( B_i \) for a given bankroll \( BR \). The Kelly utility may be written as
\[ J = \sum_i c_i \text{Ex}[\log(1 + f_i U_i)] \]  

(34)

where \( \text{Ex}[...] \) is the expectation operator, and \( \{U_i\} \) is the set of possible outcomes at count \( i \), that is \( \{U_i\}=\{-2,-1,-0.5,...\} \), and \( f_i \) is the bankroll fraction such that

\[ B_i = BR \cdot f_i. \]  

(35)

Expanding the log function using

\[ \log(1 + x) \approx x - \frac{x^2}{2} + \ldots \]

gives the usual approximation for \( J \),

\[ J = \sum_i c_i \left[ f_i \cdot \text{Ex}[U_i] - \frac{f_i^2}{2} \cdot \text{Ex}[U_i^2] \right]. \]  

(36)

But we have

\[ e_i = \text{Ex}[U_i] \]
\[ v_i = \text{Ex}[U_i^2] - (\text{Ex}[U_i])^2 \approx \text{Ex}[U_i^2] \]

and so

\[ J = \sum_i c_i \left[ f_i \cdot e_i - \frac{f_i^2}{2} \cdot v_i \right] \]  

(37)

or

\[ J = ev[f] - \frac{1}{2} var[f] \]  

(38)

where the definitions Eqn(5) have been used.

The variance is a very good approximation for the second moment for the game of Blackjack. If one chooses to use \( \text{Ex}[U_i^2] \) instead of \( v_i \) it makes almost no difference, since \( e_i^2 \) is much less than \( v_i \). Now \( f_i \) is a dimensionless quantity, but is not a betting spread as defined in Eqn(3), since there is no constraint on the minimum value.

Define a Kelly Spread as

\[ f_i = 0 \quad \forall i \in \{I_{Wong}\} \]
\[ f_i = f_{\text{min}} > 0 \quad \forall i \in \{I_{\text{Min}}\} \]
\[ f_i = M \cdot f_{\text{min}} \quad \forall i \in \{I_{\text{Max}}\} \]
\[ f_{\text{min}} < f_i < M \cdot f_{\text{min}} \quad \forall i \notin \{I_{\text{Inter}}\} \]  

(39)

So again we are presented with a well defined problem:

For a given value of \( M \), specified zero set \( \{I_{\text{Wong}}\} \), and requirement \( ev[f] > 0 \), find the optimal Kelly spread \( f_{\text{opt}}^{\text{opt}} \), which minimises \( J[f] \). Again, without loss of generality, it is possible to remove the set \( \{I_{\text{Wong}}\} \) from the ordered set of values \( \{\text{Game}\} \). In which case, \( f_i \) is now bounded below by \( f_{\text{min}} \) and above by \( M \cdot f_{\text{min}} \). Following a similar procedure to Section 2, we seek to find a solution which satisfies the constraints imposed by Eqn(39). Differentiating \( J \) with respect to \( f_i \).
\[
\frac{dJ}{df_i} = c_i \cdot \left[ e_i - f_i \cdot v_i \right]
\]  

(40)

and setting the result to zero gives the familiar Kelly condition,

\[
f_i = \frac{e_i}{v_i}.
\]  

(41)

Choose a Kelly spread of the form

\[
\begin{align*}
  f_i &= f_{\text{min}} & & i \leq T_i \\
  f_i &= \frac{e_i}{v_i} & & T_i + 1 \leq i \leq T_M - 1 \\
  f_i &= M \cdot f_{\text{min}} & & i \geq T_M.
\end{align*}
\]  

(42)

Then, using the notation previously defined,

\[
J = f_{\text{min}} \cdot ev_+[T_i] + M \cdot f_{\text{min}} \cdot ev_+[T_M] - \frac{f_{\text{min}}^2}{2} \cdot \text{var}_+[T_i] - \frac{f_{\text{min}}^2}{2} \cdot M^2 \cdot \text{var}_+[T_M] + \frac{1}{2} D[T_i, T_M] 
\]  

(43)

where the final \(D[T_i, T_M]/2\) has received a contribution from both the terms in Eqn(37). Now differentiate again to fix the value of \(f_{\text{min}}\).

\[
\frac{dJ}{df_{\text{min}}} = \left[ ev_-[T_i] + M \cdot ev_+[T_M] \right] - f_{\text{min}} \cdot \left[ \text{var}_-[T_i] + M^2 \cdot \text{var}_+[T_M] \right] = 0
\]  

(44)

which when solved for \(f_{\text{min}}\) gives

\[
F_{\text{min}} = \frac{ev_-[T_i] + M \cdot ev_+[T_M]}{\text{var}_-[T_i] + M^2 \cdot \text{var}_+[T_M]}.
\]  

(45)

However, it must be checked that the value of \(f_{\text{min}}\) is consistent with Eqn(42). So we must have

\[
\begin{align*}
  f_{\text{min}} &< \frac{e_i}{v_i} & & \forall \ T_i + 1 \leq i \leq T_M - 1 \\
  M \cdot f_{\text{min}} &> \frac{e_i}{v_i} & & \forall \ T_i + 1 \leq i \leq T_M - 1
\end{align*}
\]

Which, by the now familiar properties of the ordered set \(\{e/v\}\), it is sufficient that

\[
\begin{align*}
  f_{\text{min}} &< \frac{e_{i+1}}{v_{i+1}} & & (46) \\
  f_{\text{min}} &> \frac{1}{M} \cdot \frac{e_{i+1}}{v_{i+1}}. & & (47)
\end{align*}
\]
Provided Eqns(46) and (47) are satisfied, we have a maximised Eqn(38), for the given values of \((T_i, T_M)\). But again, we do not know if this is the true maximum of \(J\), given the constraints of the problem. This time, we are attempting to maximise \(J\), not minimise \(n_a\), however the same principles hold. We require that all variations of \(J\), about the solution, act to decrease \(J\), that is we require \(\delta J < 0\). Also, we require that where the derivative vanishes, the second derivative must be negative, in order for the solution to be a maximum.

**Case(1):** \(i \leq T_1\)

Since \(f_i = f_{\text{min}}\), we must have \(\delta f_i > 0\), since if \(\delta f_i < 0\), we would require \(f_i < f_{\text{min}}\). Using Eqn(40)

\[
\delta J = c_i \left[ e_i - f_{\text{min}} \cdot v_i \right] \cdot \delta f_i < 0,
\]

which can be solved to give

\[
f_{\text{min}} > \frac{e_i}{v_i} \quad \forall \ i \leq T_1
\]

and it is again sufficient if

\[
f_{\text{min}} > \frac{e_i}{v_i}.
\]

**Case(2):** \(i \geq T_M\)

In this case \(f_i = M f_{\text{min}}\), we must have \(\delta f_i < 0\), which gives

\[
\delta J = c_i \left[ e_i - M \cdot f_{\text{min}} \cdot v_i \right] \cdot \delta f_i < 0
\]

and so

\[
f_{\text{min}} < \frac{1}{M} \cdot \frac{e_i}{v_i} \quad \forall \ i \geq T_M
\]

or

\[
f_{\text{min}} < \frac{1}{M} \cdot \frac{e_M}{v_M}.
\]

**Case(3):** \(T_1 + 1 \leq i \leq T_M - 1\)

In this case, differentiating Eqn(40) with respect to \(f_i\) gives

\[
\frac{d^2 J}{df_i^2} = - c_i \cdot v_i < 0
\]

which is always true. By combining Eqns(46)-(49), we obtain the final conditions

\[
\frac{e_{r_i}}{v_{T_i}} < f_{\text{min}} < \frac{e_{r_i+1}}{v_{T_i+1}},
\]

and

\[
\frac{1}{M} \cdot \frac{e_{T_a - 1}}{v_{T_a - 1}} < f_{\text{min}} < \frac{1}{M} \cdot \frac{e_M}{v_M}.
\]
Note that Eqns(51) and (52) impose a 'continuity' condition on the Kelly spread Eqn(42), which allows the optimal spread to be written as

\[ f_i = \begin{cases} f_{\text{min}} \cdot v_i, & \frac{e_i}{v_i} \leq f_{\text{min}} \\ M \cdot f_{\text{min}} \cdot v_i, & \frac{e_i}{v_i} \geq M \cdot f_{\text{min}} \end{cases} \]

otherwise. (53)

The Yamashita method can therefore be seen as a way to progressively eliminate the Kelly spreads which do not satisfy Eqns(51) and (52), to arrive at the final spread which satisfies Eqn(53).

Now the astute reader may have begun to notice many similarities between this problem and the one solved in Section 2. In particular, by the Equations (42),(45),(51) and (52), \( f_i \) can be expressed in terms of a betting spread \( b_i \) of the form

\[ f_i = f_{\text{min}} \cdot b_i \]

where \( ekb(b) = \frac{1}{f_{\text{min}}} \). (55)

**Theorem:** The Kelly scheme \( f_i \) which maximises Eqn(38), can be written in terms of an optimal (in the \( n_o \) sense) betting spread \( b_i \) where \( ekb(b) = 1/f_{\text{min}} \).

The optimal Kelly betting scheme \( B_i \) can now be written down as

\[ B_i = \text{Bank} \cdot f_i \]

\[ = \text{Bank} \cdot f_{\text{min}} \cdot b_i = \left[ \frac{\text{Bank}}{ekb(b)} \right] \cdot b_i, \]

or in other words, the monetary scaling factor \( B_u \), is given by

\[ B_u = \frac{\text{Bank}}{ekb(b)} \]. (57)

One therefore sees that in the Kelly context, \( ekb(b) \) can be interpreted as the Kelly bankroll for which the optimal minimum bet is 1 unit.

Now as with all mathematical problems, the correspondence in Eqns(54) and (55) can not be a coincidence. Returning to Eqn(37), it is possible to rewrite the expression for \( J \) in terms of a scalar Kelly fraction \( f \), and a betting spread \( b_i \),

\[ J = \sum_i c_i \left[ f \cdot b_i \cdot e_i - f^2 \cdot \frac{b_i^2}{2} \cdot v_i \right], \]

or

\[ J = f \cdot ev(b) - \frac{1}{2} f^2 \cdot var(b), \]
where now since \( b_i \) is constrained between 1 and \( M \), the extra degree of freedom is taken up by the Kelly fraction \( f \). Differentiating with respect to \( f \) and setting the result to zero,

\[
\frac{dJ}{df} = ev[b] - f \cdot var[b] = 0 ,
\]
gives

\[
f = \frac{ev[b]}{var[b]} = \frac{1}{ekb[b]} .
\] (60)

Note that this is the result given by 'Grimy Fellow', which shows that it is possible to compute the Kelly fraction for any spread whatsoever. One now has the betting scheme

\[
B_i = \text{Bank} \cdot f \cdot b_i = \left[ \frac{\text{Bank}}{ekb[b]} \right] \cdot b_i ,
\] (61)

which is the Kelly betting scheme for the bankroll \( \text{Bank} \) and arbitrary spread \( b_i \). There may be many reasons why a player may choose a non-standard spread \( b_i \), such as a flat spread, and Eqns(60) and (61) provide the correct Kelly betting scheme for such a method. Substituting Eqn(60) back into Eqn(59), gives

\[
J[b] = \frac{1}{2} \cdot \frac{ev^2[b]}{var[b]} = \frac{1}{2n_o[b]} .
\] (62)

It can be immediately seen that the spread which now maximises \( J[b] \), under the conditions specified, must by Eqn(62), also minimise \( n_o[b] \). This now suffices to show complete equivalence between any method which minimises \( n_o[b] \), and the Yamashita method which solves for the value of \( f \) which maximises \( J[f] \). However, the Yamashita method is pure Kelly from the outset, and as such is inapplicable to any betting method which does not use Kelly as a basis. Eqn(62) shows that while the minimisation of \( n_o[b] \) is a component of the Kelly scheme, it is in itself a separate problem. It is possible to use the optimal spread \( b_i \) in a fixed betting scheme, without any reference to Kelly at all, and as such, the \( n_o[b] \) minimisation method has much wider applicability.

**Section 4 - Fixed Betting Schemes**

A fixed betting scheme differs from a Kelly scheme, in that the betting unit is not varied as the bankroll changes. As a consequence, the main consideration for a fixed scheme is Risk of Ruin, where there is a finite chance that the bankroll may reach zero. In this section, I will show that the optimal betting spread \( b_i \), is precisely the spread which gives the fastest linear average growth, for a given initial bankroll and fixed risk of ruin.

Firstly, for the fixed betting scheme \( B_i \) defined by the product of a dollar betting unit \( B_u \) with a betting spread \( b_i \)

\[
B_i = B_u \cdot b_i ,
\] (63)

it is possible to define the Betting functionals,

\[
EV[B] = \sum_i c_i \cdot B_i \cdot e_i = B_u \cdot ev[b] ,
\] (64)

\[
\text{VAR}[B] = \sum_i c_i \cdot B_i^2 \cdot v_i = B_u^2 \cdot \text{var}[b] ,
\] (65)

and the derived quantities
\[ EKB[B] = \frac{\text{VAR}[B]}{\text{EV}[B]} = B_u \cdot ekb[b] \quad (66) \]
\[ N_o[B] = \frac{\text{VAR}[B]}{\text{EV}^2[B]} = n_o[b] \quad (67) \]

Note from Eqns(64) and (65), \( N_o[B] = n_o[b] \), which shows that \( N_o[B] \) is independent of \( B_u \), therefore the quantity \( n_o[b] \) is an invariant property of the system.

As a short aside, for a Kelly betting scheme of the form Eqn(58), we have \( B_u = \frac{\text{Bank}}{ekb[b]} \) and so
\[ \text{EV}[B] = \frac{\text{Bank}}{n_o[b]}, \]
\[ \text{VAR}[B] = \frac{\text{Bank}^2}{n_o[b]}, \]
\[ EKB[B] = \frac{\text{Bank}}{\text{EV}^2[B]}, \]
which shows that at the minimum of \( n_o[b] \), \( \text{EV}[B] \) as a fraction of bankroll is maximised, and the system functional \( K[B] \) is equal to the bankroll itself.

The historical origin of \( N_o[B] \) resulted from the observation that if a player is unlucky enough to be losing at \( Q \) standard deviation below expectation, then the profit as a function of rounds \( N \) is given by
\[ P(N) = \text{EV}[B] \cdot N - Q \cdot \text{SD}[B] \cdot \sqrt{N}, \quad (69) \]
where \( \text{VAR}[B] = \text{SD}^2[N] \). This function initially drops below zero, and does not reach zero again until
\[ N_o[Q, B] = Q^2 \cdot \frac{\text{VAR}[B]}{\text{EV}^2[B]} \quad (70) \]

Hence, \( N_o[B] \) can be seen as the number of rounds required to overcome one standard deviation of ‘bad luck’. This is the origin of the interpretation of \( N_o[B] \) as a number of rounds, and can be seen as an index which measures how long the ‘long run’ is for a particular game. In this sense, the minimisation of \( N_o[B] \) can be seen to have nothing whatsoever to do with Kelly betting.

This however, only suggests that it is a good idea to minimise \( N_o[B] \) for a given betting scheme. Using the concept of risk of ruin, the analysis can be made much more concrete. The ROR equation can be made more general for various bankroll goals, and other requirements, but I will use the simple, infinite win goal form, given in Blackjack for Blood, by Bryce Carlson,
\[ R = \frac{1 + (\text{EV}[B]/\text{SD}[B])^{-B_u}}{1 - (\text{EV}[B]/\text{SD}[B])} \quad (71) \]
where \( \text{Bank} \) is the starting bankroll. The exponent \( \text{Bank}/\text{SD}[B] \) deserves some examination. The system functional \( EKB[B] \) has units of money, and as such can serve as a reference point. If the starting bankroll \( \text{Bank} \) is equal to
\[ \text{Bank} = \alpha \cdot EKB[B], \quad (72) \]
then
\[ \frac{\text{Bank}}{\text{SD}[B]} = \frac{\alpha \cdot EKB[B]}{\text{EV}[B] \cdot \sqrt{N_o[B]}} \]
\[
\begin{align*}
\frac{\alpha \cdot B_u \cdot e^{kb[b]}}{B_u \cdot e^{v[b]} \cdot \sqrt{n_0[b]}} &= \alpha \cdot \sqrt{n_0[b]}, \\
(n_0[b]) \text{ and } &\alpha \text{ alone}, \\
R &= \left[1 + \sqrt{n_0[b]} \frac{\alpha \sqrt{n_0[b]}}{1 - n_0[b]}\right]^{-\alpha} \equiv Y^{-\alpha}. \\
Y &= \left[1 + \sqrt{n_0[b]} \frac{\sqrt{n_0[b]}}{1 - n_0[b]}\right],
\end{align*}
\]

where Eqns(5), (6) and (11) have been used throughout. We can then rewrite Eqn(71) in terms of \(n_0[b]\) and \(\alpha\) alone,

\[
R = \frac{1 + \sqrt{n_0[b]}}{1 - n_0[b]} \equiv Y^{-\alpha}.
\]

For most games \(n_0[b]\) is of the order of 10,000 to 100,000, so that \(\sqrt{n_0[b]}\) is much greater than 100 and we can use the approximation,

\[
(1 + \frac{\alpha}{\sqrt{n_0[b]}}) = (1 - \frac{\alpha}{\sqrt{n_0[b]}}) = e \approx 2.718...
\]

to show that

\[
Y \approx e^2 + O\left(\frac{1}{\sqrt{n_0[b]}}\right),
\]

and so \(Y\) is for most practical purposes independent of \(n_0[b]\). So we get the very simple form for the risk of ruin,

\[
R = e^{-2\alpha},
\]

where alpha is simply the ratio between the bankroll \(Bank\) and the system functional \(EKB[B]\). This is not an entirely new result, Patrick Sileo has previously derived such an expression. The main difference here is in the interpretation of the quantities involved, and their relation to the betting spread, optimal or otherwise. One immediate result of Eqn(76) is the often quoted figure that the risk of ruin for a fixed spread, where the bankroll is given by the ‘Equivalent Kelly Bankroll’, \(EKB[B]\), the name given in Eqns(11) and (66), is equal to

\[
R(\alpha = 1) = e^{-2} = 13.5\%.
\]

Now given a fixed risk of ruin \(R\), and initial bankroll \(Bank\), the median bankroll after \(N\) rounds is

\[
Bank(N) = EV[B] \cdot N + Bank
\]

\[
= B_u \cdot e^{v[b]} \cdot N + Bank
\]

but from Eqn(66) and (72) we have

\[
B_u = \frac{Bank}{\alpha \cdot e^{kb[b]}},
\]

and so

\[
Bank(N) = \frac{1}{\alpha} \cdot Bank \cdot \frac{N}{n_0[b]} + Bank
\]
to illustrate the above theory. Here is an example of a six deck AC type game, using the Zen count. The paper has been rather theoretical so far, and it is now time to provide some concrete examples to interpret the above theory. Here is an example of a six deck AC type game, using the Zen count. The raw data used to define the Game is

$$\text{Bank} = \left(1 + \frac{N}{\alpha \cdot n_i(b)}\right) \cdot \text{Bank} \cdot \alpha = \ln\left(1/\sqrt{R}\right).$$

Equation (80) shows that for a fixed initial bankroll \textit{Bank}, and a given risk of ruin \textit{R}, the spread \textit{b} which minimises \textit{n}_i(\textit{b}) produces the fastest linear growth rate of any fixed betting scheme.

\section{Section 5 - Examples}

This paper has been rather theoretical so far, and it is now time to provide some concrete examples to illustrate the above theory. Here is an example of a six deck AC type game, using the Zen count. The raw data used to define the Game is
and the optimal spreads for

1) 1 to 10, play only \( i \) greater than or equal to \(-2\): \( b_i^{\text{Wong}} \)
2) 1 to 10, play all: \( b_i^{\text{Alt}} \)

are also given for later reference. For this game, the minimum possible value for \( T_i \) for any spread is 0, since \( e_0 < 0 \), and \( e_T > 0 \).

As outlined in Section 2, the problem reduces to finding the value of \((T_1,T_M)\) for which Eqns(31) and (32) are satisfied. Equation(20) shows that \( ekb(b) \) is a function of \( T_1 \) and \( T_M \), and the basic parameters of the Game. It is therefore only a matter of computing \( ekb(T_1,T_M) \) for each value \((T_1,T_M)\), where \( T_1 < T_M \) and checking to see if \( 1/ekb(T_1,T_M) \) falls between the appropriate \( e/v \). Define

\[
k(i) \equiv \frac{v_i}{e_i},
\]

then Eqns(31) and (32) become

\[
\frac{1}{k(T_1)} \leq \frac{1}{ekb(T_1, T_M)} \leq \frac{1}{k(T_1 + 1)} \tag{80}
\]

and

\[
\frac{1}{M} \cdot \frac{1}{k(T_M - 1)} \leq \frac{1}{ekb(T_1, T_M)} \leq \frac{1}{M} \cdot \frac{1}{k(T_M)} \tag{81}
\]

It is a condition that \( k(T_1 + 1) > 0 \), but \( k(T) \) may be negative, so while we can rewrite Eqns(80) and (81) as

\[
k(T_1 + 1) \leq ekb(T_1, T_M) \leq k(T_1) \tag{82}
\]
and
\[ M \cdot k(T_M) \leq ekb(T_i, T_M) \leq M \cdot k(T_M - 1) \]  \hspace{1cm} (83)

we require the special case, that if \( k(T_i) < 0 \), the right hand inequality of Eqn(82) will be deemed to be satisfied. By tabulating the values of \( ekb(T_i, T_M), k(T_i) \) and \( k(T_M) \), the values satisfying Eqns(82) and (83) and hence the optimal spread by Eqn(33), may be found by inspection.

**Case(1) : Play only i=-2.**

<table>
<thead>
<tr>
<th>( T_i )</th>
<th>( k(T_i) )</th>
<th>( T_M )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
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<td>&gt;Start</td>
<td>1514.9</td>
<td>1134.8</td>
<td>914.8</td>
<td>773.9</td>
<td>677.9</td>
<td>611.8</td>
<td>568.3</td>
<td>543.5</td>
<td>539.0</td>
<td>556.1</td>
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<td>--</td>
<td>--</td>
<td>1134.6</td>
<td>915.6</td>
<td>775.5</td>
<td>678.0</td>
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<td>573.2</td>
<td>550.1</td>
<td>547.8</td>
<td>567.7</td>
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</tr>
</tbody>
</table>

| \( 10 \cdot k(T_M) \) | -6142.4 | 10729.3 | 3104.1 | 1800.2 | 1262.3 | 970.5 | 781.5 | 654.9 | 557.3 | 492.6 | 438.6 |
| \( T_M \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

The procedure is as follows:

1) Start at the top left hand corner.
2) Move across the table until \( ekb(T_i, T_M) \) is greater than the value in the bottom row \( M^*k(T_M) \).
3) Check that \( ekb(T_i, T_M) \) is less than the preceding value in the bottom row \( M^*k(T_M-1) \).
4) Now check to see if \( ekb(T_i, T_M) \) is less than the value at the left \( k(T_i) \) or \( k(T_i) < 0 \).
5) Check that \( ekb(T_i, T_M) \) is greater than the next value at the left \( k(T_i+1) \).
6) If any of the preceding stages are unable to be satisfied, proceed to the next row and repeat.
7) If all (2)-(5) are satisfied, we have found \( (T_i, T_M) \) and \( ekb(T_i, T_M) \) and are done.

For the example above:

\( T_i = 0 \) : \( ekb(0, 9) = 539.0 \) satisfies (2), (3), and (4), but fails (5) since \( k(1) = 1072.9 > 539.0 \).
\( T_i = 1 \) : \( ekb(1, 9) = 547.8 \) satisfies (2), (3), (4), and satisfies (5) since \( k(2) = 310.4 < 547.8 \).

Note that for \( T_i = 2 \), the value of \( ekb(2, T_M) \) is greater than \( k(2) = 310.4 \) for all \( T_M \), making it impossible to satisfy condition (4). So the optimal value of \( (T_i, T_M) \) is \((1,9)\) and \( ekb(1,9) = 547.8 \) is the optimal spread parameter. Looking at the data table above, we can see the optimal spread \( b_{k_{Wong}} \) which is derived from this value of \( ekb[b] \), using Eqn(33). Using the simple iterative method described at the end of Section 2, we get:

<table>
<thead>
<tr>
<th>ev</th>
<th>sd</th>
<th>ekb</th>
<th>ev/sd</th>
<th>N0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.019673</td>
<td>3.2827</td>
<td>547.78</td>
<td>0.59927%</td>
<td>27845</td>
</tr>
</tbody>
</table>

and so confirms the exact approach used above.
**Case(2) : Play All:**

<table>
<thead>
<tr>
<th>$T_j$</th>
<th>$k(T_j)$</th>
<th>$\text{ekb}(T_j, T_M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-614.2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1072.9</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>310.4</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>180.0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>126.2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>97.1</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>78.2</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>65.5</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>55.7</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>49.3</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>43.9</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$T_M$</th>
<th>$\text{ekb}(T_j, T_M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Again following the procedure outlined above

$T_j = 0$ : $\text{ekb}(0, 6) = 881.9$ satisfies (2), (3), and (4), but fails (5) since $k(1) = 1072.9 > 881.9$.
$T_j = 1$ : $\text{ekb}(1, 6) = 883.9$ satisfies (2), (3), (4), and satisfies (5) since $k(2) = 310.4 < 883.9$.

Again, for $T_j = 2$, all the values of $k(T_j)$ are too small to allow (4) to be satisfied. So we have $(T_j, T_M)$ as (1,6) and the spread parameter $\text{ekb}(1, 6) = 883.9$. Using this value in Eqn(33), generates the full spread as given in the table above. Again as a check, running this case through the iterative algorithm gives,

<table>
<thead>
<tr>
<th>$\text{ev/SD}$</th>
<th>$\text{ekb}$</th>
<th>$\text{N}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.019959</td>
<td>883.94</td>
<td>0.47519%</td>
</tr>
<tr>
<td>4.2003</td>
<td></td>
<td>44287</td>
</tr>
</tbody>
</table>

which again confirms the result of the direct calculation.

Now for these examples to be of any use, we must choose a betting method.

**Kelly Method**

In a previous post, the example was given of a Kelly bettor with a bankroll of $\text{Bank} = $5000. Using Eqn(54), we can immediately compute the scaling factor $B_w$.

$$B_w = \frac{5000}{547.8} = 9.13 \text{ for the Wong case,}$$
$$B_a = \frac{5000}{883.9} = 5.66 \text{ for the play all case.}$$

The Kelly betting scheme is then given by

$$B_{W} = B_w * h_{W}$$  $$B_{A} = B_a * h_{A}$$  \hspace{1cm} (84)

with the values of $B_{W}$ and $B_{A}$ given in the data table above.

**Fixed Betting Method**

In this example, we will keep the initial bankroll C=$5000, and choose a risk of ruin , $R=5\%$, which gives alpha = 1.498. This means that the scaling factor A is then from Eqn(74),

$$B_{W} = \frac{5000}{(1.498*547.8)} = 6.09$$
$$B_{A} = \frac{5000}{(1.498*883.9)} = 3.78$$
and again the betting schemes may be computed as in Eqn(84).

As a test of the validity of the above analysis, a simulation was set up with the same game settings as that which generated the above table. For a given betting scheme and bankroll, sets of trials were run, where if the bankroll was doubled a 'win' was recorded, and if the bankroll was lost, a 'ruin' was recorded. In either case, the system is reset and another trial begun. This gives the risk of ruin before doubling, which is only slightly less than the infinite win formula given above. We have four cases, two with the Wonging scheme and two with the play all scheme. For each scheme, two bankrolls are used, the $5000 bankroll specified to give a 5% risk of ruin, and the K[B] bankroll, designed to test Eqn(73). In all cases, 100 trials were done and the risk of ruin, the average number of hands to ruin, or to win is computed.

Results:

<table>
<thead>
<tr>
<th>Spread</th>
<th>Bank</th>
<th>ROR</th>
<th>AvHands(W)</th>
<th>AvHands(R)</th>
<th>$N_0[B]$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_{\text{Wong}}$</td>
<td>$5000$</td>
<td>4.0%</td>
<td>75887</td>
<td>36073</td>
<td>27845</td>
<td>1.498</td>
</tr>
<tr>
<td>$b_{\text{Wong}}$</td>
<td>$3340$</td>
<td>12.0%</td>
<td>47114</td>
<td>22351</td>
<td>27845</td>
<td>1.000</td>
</tr>
<tr>
<td>$b_{\text{All}}$</td>
<td>$5000$</td>
<td>3.0%</td>
<td>127251</td>
<td>93909</td>
<td>44287</td>
<td>1.498</td>
</tr>
<tr>
<td>$b_{\text{All}}$</td>
<td>$3340$</td>
<td>12.0%</td>
<td>68365</td>
<td>62653</td>
<td>44287</td>
<td>1.000</td>
</tr>
</tbody>
</table>

It can be seen that, while the ROR figures are approximate for the small number of trials, the figures are consistent with the above analysis. It is also apparent that the average doubling time is proportional to $N_0[B]$, as it must be said, is the average ruin time. This may be related to the concept of trip-risk which has been discussed by Bryce Carlson.

**Conclusion**

In this article, it has been demonstrated that the basis of all forms of optimal betting, resides with the optimal betting spread $b_i$. It has been shown that through the minimisation of the functional $n_0[b]$, the optimal spread $b_i$ may be computed in a way which is independent of any betting method which may be used. Further, a method has been given which allows direct calculation of the functional $ekb[b]$, which defines the solution of the problem. The explicit Kelly scheme was examined, and shown to produce the same solution, provided the minimum Kelly fraction $f_{\text{min}}$ is identified with $1/ekb[b]$. However, it was also demonstrated that for a fixed betting method, the optimal spread produces the fastest mean linear growth, for a given bankroll and risk of ruin. It was shown in this case that the ratio $\text{Bank}/(B_u ekb[b])$, where $\text{Bank}$ is the bankroll and $B_u$ is the betting factor, determines the risk of ruin.

So in conclusion, the concept of an optimal betting spread stands apart from the betting method used. Once the betting method has been chosen, be it Kelly or fixed, this then provides the interpretation of the functional $ekb[b]$. While the optimal betting scheme can be directly calculated from within a Kelly system, this must be seen a special case of a more general problem.